Mathematical Finance
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Assignment 2

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

Approximating stopping times

Let τ be an $\mathbb{F}$–optional time. Define for any integer $n \in \mathbb{N}^\star$ the following random times, for any $\omega \in \Omega$

$$\tau_n(\omega) := \begin{cases} \tau(\omega), \text{ if } \tau(\omega) = +\infty, \\ \displaystyle\sum_{k=1}^{+\infty} \frac{k}{2^n} \mathbf{1}_{\{(k-1)/2^n \leq \tau(\omega) < k/2^n\}}. \end{cases}$$

Show that $(\tau_n)_{n \in \mathbb{N}^\star}$ is a non-increasing sequence of $\mathbb{F}$–stopping times, which converges to τ, and such that for any set $A \in \mathcal{F}_{\tau+}$, we have $A \cap \{\tau_n = k/2^n\} \in \mathcal{F}_{k/2^n}$, for any integers $(n,k) \in \big(\mathbb{N}^\star\big)^2$.

Local boundedness and filtrations

Let U be a non-negative and unbounded random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$X_t := U \mathbf{1}_{\{t>1\}},\ t \geq 0,$$

and let $\mathbb{F} := \mathbb{F}^X$.

1) Show that $\mathbb{F}$ is not right-continuous and that a random time τ is an $\mathbb{F}$–stopping time if and only if τ is a deterministic number in $[0,1]$, or $\tau = f(U)$ for some Borel-measurable map $f : \mathbb{R} \longrightarrow (1, +\infty)$.

2) Show that X is not $(\mathbb{F}, \mathbb{P})$–locally bounded.

Émery topology and change of measure

Show that the $(\mathbb{F}, \mathbb{P})$–Émery topology is invariant under an equivalent change of measure.

Completeness and the Émery topology

We let $\mathcal{S}_b(\mathbb{F}, \mathbb{P})$ be the space of bounded, simple $\mathbb{F}$-predictable processes, and we assume that $\mathbb{F}$ satisfies the usual conditions.

1) Show that the space of càdlàg and $\mathbb{F}$-adapted processes, as well as the space of càglàd and $\mathbb{F}$-adapted processes are complete under the $(\mathbb{F}, \mathbb{P})$–Émery topology (you may use here the result of Proposition 4.6.5 and Theorem 4.6.2 from the lecture notes).

2) Show that if X is càdlàg process, the following are equivalent

   1. the map J_X from $\mathcal{S}_b(\mathbb{F}, \mathbb{P})$ to the set of càdlàg and $\mathbb{F}$-adapted processes defined by

   $$J_X(\xi) := \int_0^\cdot \xi_s \mathrm{d}X_s,$$

   is continuous with respect to $\mathbb{P}$–ucp convergence on both spaces;

   2. for every $t \in [0, +\infty)$, the mapping I_{X^t} from $\mathcal{S}_b(\mathbb{F}, \mathbb{P})$ to $\mathbb{L}^0(\mathbb{R}, \mathcal{F})$ with $I_{X^t}(\xi) := J_X(\xi)_t$, is continuous with respect to the uniform norm metric on $\mathcal{S}_b(\mathbb{F}, \mathbb{P})$ and convergence in $\mathbb{P}$-probability on $\mathbb{L}^0(\mathbb{R}, \mathcal{F})$.